Stability of the Schwarzschild Interior in Loop Quantum Gravity

Christian G. Böhmer[1,*] and Kevin Vandersloot[2,†]

[1]Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, UK

[2]Institute of Cosmology & Gravitation, University of Portsmouth, Portsmouth PO1 2EG, UK

(Dated: August 14, 2008)

In recent work, the Schwarzschild interior of a black hole was investigated, incorporating quantum gravitational modifications due to loop quantum gravity. The central Schwarzschild singularity was shown to be replaced by a Nariai type universe. In this brief report we show that this interior solution is stable with respect to small perturbations, in contrast to the classical Nariai universe.

PACS numbers: 04.60.Pp, 04.70.Bw, 98.80.Qc, 03.65.Sq

In a recent article [1], the interior of a Schwarzschild black hole was analyzed in the framework of loop quantum gravity (LQG – for reviews see [2, 3, 4]). In the analysis it was shown that loop quantum effects can cure the central singularity and replace it with a Nariai type universe with a Planck scale two-sphere radius. This would seem to imply that infalling matter would settle in a finite Planckian radius as opposed to collapsing to the classical singularity, though a more rigorous analysis of the loop effects would be needed to answer such a question. However, concerns were raised in [5] that the Nariai solution of classical general relativity is unstable to certain perturbations with the instabilities leading to a Schwarzschild-de Sitter solution.

In this brief report, we show that the Nariai solution in the interior is stable owing to the loop quantum effects in contrast to the classical situation. We perform a perturbative analysis about the Nariai solution using the loop quantum equations of motion. The linearized perturbation equations can be solved analytically and we show that the perturbations decay if evolved towards the center thereby establishing the stability of our solution. Note that the issue of stability we investigate is separate from the stability studied in [6]. There, the stability of the underlying discrete quantum theory is discussed in the context of various quantization ambiguities of the theory. The analysis of [6] is relevant to this work in that it supports the choice of δ parameters that we use in the effective Hamiltonian (6).

We briefly recall the equations describing the Schwarzschild interior in the LQG framework. For more details the reader is referred to [7, 8]. The interior has been studied in LQG in [9, 10, 11]. For more details, the reader is referred to those articles. The interior metric is of the form

$$ds^2 = -N(t)^2 dt^2 + g_{xx}(t)\, dx^2 + g_{\Omega\Omega}(t)\, d\Omega^2 \tag{1}$$

with $N(t)$ being the freely specifiable lapse function and $d\Omega^2$ representing the unit two-sphere metric. The classical Schwarzschild interior solution is given by

$$N(t)^2 = \left(\frac{2m}{t} - 1\right)^{-1}, \quad g_{xx}(t) = \left(\frac{2m}{t} - 1\right), \quad g_{\Omega\Omega}(t) = t^2 \tag{2}$$

for t in the range $t \in [0, 2m]$, where m is the mass of the black hole. Here, $t = 0$ corresponds to the classical singularity and $t = 2m$ being the horizon. Loop quantum gravity uses a separate set of variables than the metric ones. The metric components are encoded in triad components p_b, p_c which are related as

$$g_{xx} = \frac{p_b^2}{|p_c|}, \quad g_{\Omega\Omega} = |p_c|\,. \tag{3}$$

Note that the physical radius of the two-sphere is determined from the $g_{\Omega\Omega}$ or p_c components. In particular, the classical singularity corresponds to a vanishing two-sphere radius where $g_{\Omega\Omega} = p_c = 0$. The LQG variables also consist of connection components b, c which are conjugate to the triad components. We focus on the triad components for interpretation since they are directly related to the metric components.

The Nariai solution [12, 13] is a spherically symmetric solution of the classical field equations of general relativity where the two-sphere radius is assumed to be constant and the model is sourced by a positive cosmological constant Λ with the possible inclusion of charge. For the case of no charge, the metric is given by

$$ds^2 = -dt^2 + \frac{1}{\Lambda}\cosh^2(\sqrt{\Lambda}t)dx^2 + \frac{1}{\Lambda}d\Omega^2 \tag{4}$$

*Electronic address: c.boehmer@ucl.ac.uk

†Electronic address: kevin.vandersloot@port.ac.uk

the form of which is equivalent to (1). The triad components are thus given by

$$p_b = \frac{1}{\Lambda}\cosh(\sqrt{\Lambda}t), \quad p_c = \frac{1}{\Lambda}\,. \tag{5}$$

The classical Nariai solutions are quantum mechanically unstable. Such instabilities lead to perturbations in the two-sphere radius p_c and the solution decays into a Schwarzschild-de Sitter black hole solution (or Reissner-Nordstrom-de Sitter for the charged case) [14, 15, 16].

The loop quantum effects are derived from a modified effective Hamiltonian [1, 17] based on the phase-space variables b, c, p_b, p_c which is given by (note that we use the improved quantum Hamiltonian of section IIIB in [1])

$$H_{\rm eff} = -\frac{N}{2G\gamma^2}\left[2\frac{\sin b\delta_b}{\delta_b}\frac{\sin \delta_c c}{\delta_c}\sqrt{p_c} + \left(\frac{\sin^2 b\delta_b}{\delta_b^2} + \gamma^2\right)\frac{p_b}{\sqrt{p_c}}\right]. \tag{6}$$

Here γ is known as the Barbero-Immirzi parameter, a real valued ambiguity parameter of LQG than can be constrained from black-hole entropy considerations. The parameters δ_b, δ_c are functions of the triad variables, the exact form given in [1]. These parameters directly measure the magnitude of the loop quantum effects. In the limit of $\delta_b, \delta_c \to 0$, the classical behavior is recovered.

Choosing the lapse function to be $N = \gamma\sqrt{p_c}\delta_b/(\sin b\delta_b)$, the complete set of equations derived from the effective Hamiltonian (6) are given by

$$\begin{aligned}
\dot{c} &= -\frac{\sin c\delta_c}{\delta_c} - \frac{1}{2}\frac{\sin \delta_b b}{\delta_c} - c\cos c\delta_c \\
&\quad + \frac{1}{2}\frac{\delta_b}{\delta_c} b\cos b\delta_b\left(1 - \frac{\gamma^2\delta_b^2}{\sin^2 b\delta_b}\right) + \frac{\gamma^2}{2}\frac{\delta_b^2}{\delta_c}\frac{1}{\sin b\delta_b} && (7)\\
\dot{p_c} &= 2p_c\cos c\delta_c && (8)\\
\dot{b} &= -\frac{1}{2}\left(\frac{\sin b\delta_b}{\delta_b} + \frac{\gamma^2\delta_b}{\sin b\delta_b}\right) - \frac{\sin c\delta_c}{\delta_b} + \frac{\delta_c}{\delta_b}c\cos c\delta_c && (9)\\
\dot{p_b} &= \frac{1}{2}\cos b\delta_b\left(1 - \frac{\gamma^2\delta_b^2}{\sin^2 b\delta_b}\right)p_b. && (10)
\end{aligned}$$

With this choice of lapse, the classical singularity would correspond to $T \to -\infty$. Furthermore, the system is a constrained type whereby the Hamiltonian should vanish, providing an additional equation of motion.

The results of a numerical solution to the equations of motion are shown in figures 1. The results indicate that p_c tends to a constant value as T tends to minus infinity while p_b grows as an exponential as in the Nariai case, see Eqs. (5). This is indicative of the space-time metric approaching a Nariai type universe as the classical singularity is approached.

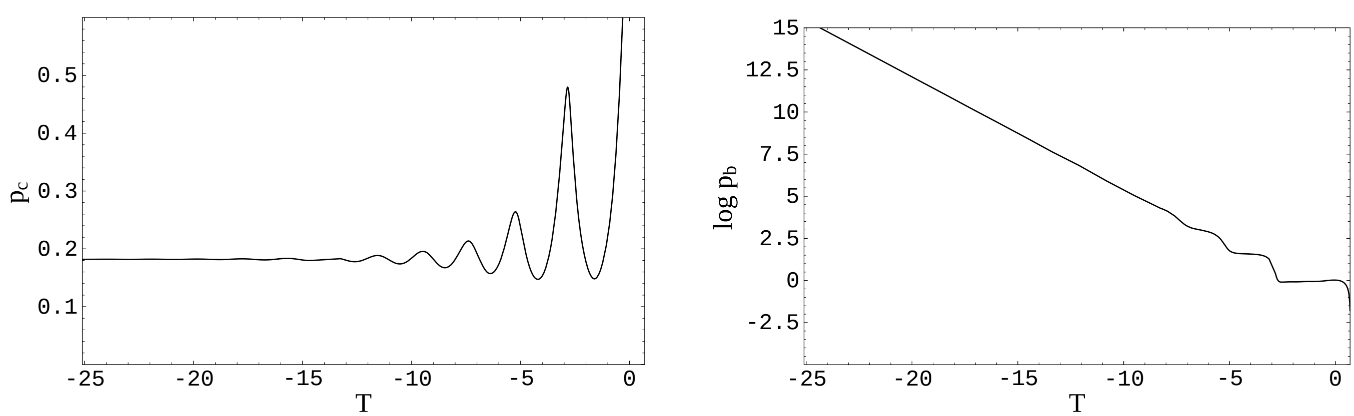


FIG. 1: In the left figure $p_c(T)$ is plotted which tends towards a constant as $T \to -\infty$. The right figure shows $p_b(T)$ which grows like $\exp(-\alpha T)$ asymptotically. This asymptotic behavior is indicative of a Nariai type metric.

In the limit of $T \to -\infty$, there is an exact asymptotic solution to the quantum equations of motion to which the solution shown in figure 1 tends to. This solution is characterized by

$$\begin{aligned}
b &= \bar{b}, & p_b &= \bar{p}_b\, e^{-\alpha T}, \\
c &= \bar{c}\, e^{-\alpha T}, & p_c &= \bar{p}_c,
\end{aligned} \tag{11}$$

where the barred quantities and α are constants. It is clear that this asymptotic solution corresponds to a Nariai type solution (5) in the large negative T limit. From the equations of motion we can determine the asymptotic values of the constants $\bar{p}_c, \bar{b}, \alpha$, and $\bar{c}/\bar{p}_b$. A root find algorithm can be used to determine the values of the constants which are given by [1]

$$\begin{aligned} \bar{b} &\approx 0.156, \qquad & \bar{p}_c &\approx 0.182\, l_{\mathrm{Pl}}^2, \qquad & \alpha &\approx 0.670 \\ \bar{c}/\bar{p}_b &\approx -2.290\, m_{\mathrm{Pl}}^2, \qquad & \bar{N} &\approx 0.689, \end{aligned} \tag{12}$$

where $\bar{N}$ is the asymptotic value of the lapse which also behaves as a constant. It is noteworthy that the fixed two-sphere radius is Planckian in length since the value of p_c approaches a constant Planckian value of $0.182 l_{\mathrm{Pl}}^2$. Furthermore this value is independent of the mass of the black hole.

Given that the classical Nariai solution is unstable to perturbations of the two-sphere radius, we can now ask the same question in the context of the loop quantum equations. We consider perturbations around the asymptotic solution (11) of the form

$$c = \bar{c} e^{-\alpha T}(1 + \varepsilon c^{(1)}(T)), \tag{13}$$

$$p_c = \bar{p}_c + \varepsilon p_c^{(1)}(T), \tag{14}$$

$$b = \bar{b} + \varepsilon b^{(1)}(T), \tag{15}$$

$$p_b = \bar{p}_b e^{-\alpha T}(1 + \varepsilon p_b^{(1)}(T)). \tag{16}$$

Now, we insert these expressions into the equations of motion and linearize the equations with respect to ε. Since we perturb about the Nariai background the zero order equations are identically satisfied. Since the four equations of motion are constrained due to the vanishing of the Hamiltonian, we are effectively left with three independent equations. This linear system of differential equations describing the perturbations then becomes

$$\frac{d}{dT}\begin{pmatrix} c^{(1)} \\ p_c^{(1)} \\ p_b^{(1)} \end{pmatrix} = \begin{pmatrix} -2.2408 & -6.4778 & 2.2408 \\ 0.5717 & 1.5708 & -0.5717 \\ 0 & 16.0296 & 0 \end{pmatrix} \begin{pmatrix} c^{(1)} \\ p_c^{(1)} \\ p_b^{(1)} \end{pmatrix}, \tag{17}$$

and is solved by

$$c^{(1)} = (4.8426 - 3.8426 e^{\delta T}\cos(\omega T) + 2.5552 e^{\delta T}\sin(\omega T)) c_0^{(1)})\xi, \tag{18}$$

$$p_c^{(1)} = (\cos(\omega T) - 0.6271 \sin(\omega T)) e^{\delta T}\xi, \tag{19}$$

$$p_b^{(1)} = (4.8426 - 3.8426 e^{\delta T}\cos(\omega T) - 4.8510 e^{\delta T}\sin(\omega T)) c_0^{(1)})\xi, \tag{20}$$

where $\omega = 3.0390$, $\delta = 0.3350$ and where ξ denotes the initial perturbation. Note that we are interested in the dynamical behavior of the perturbation close to the classical singularity, located at $T = -\infty$. Therefore, one can immediately conclude that the perturbations damp off. A plot of the perturbation behavior is shown in figure 2, showing the damped oscillatory behavior for $T \to -\infty$.

We have thus shown that the Nariai behavior near the classical singularity is a stable solution. The quantum interior will not decay into a separate Schwarzschild-de Sitter universe in contrast to the the classical Nariai universe. However, the most important issue of how the loop modifications affect a real inhomogeneous collapse scenario remains an open problem.

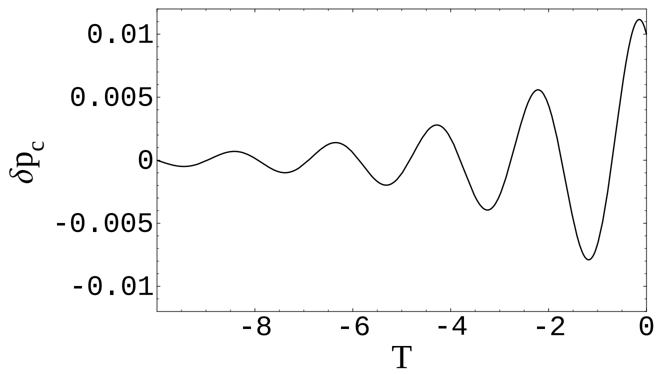
Acknowledgments

KV is supported by the Marie Curie Incoming International Grant M1F1-CT-2006-022239.

[1] C. G. Böhmer and K. Vandersloot. Loop Quantum Dynamics of the Schwarzschild Interior. *Phys. Rev.*, **D76** 104030, 2007.

[2] A. Ashtekar and J. Lewandowski. Background independent quantum gravity: A status report. *Class. Quant. Grav.*, 21:R53, 2004.

[3] C. Rovelli. *Quantum Gravity.* CUP, Cambridge, 2004.



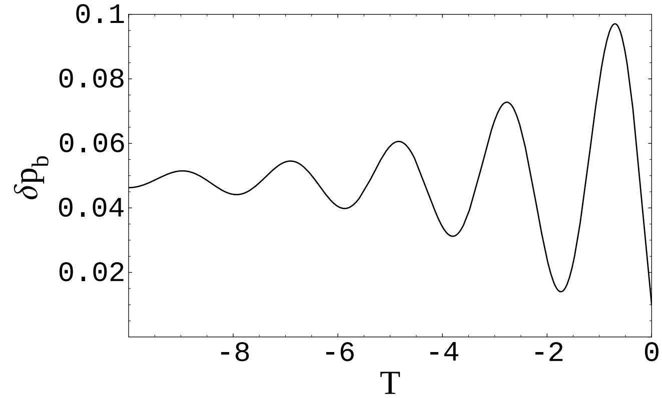


FIG. 2: In the left figure we plotted the perturbation δp_c as a function of T. We see that the amplitude of the initial perturbation $\delta = 0.01$ decreases towards the location of the classical singularity and eventually vanishes. The right figure show the behavior of the perturbation δc which also shows a damped oscillation.

[4] T. Thiemann. *Modern canonical quantum general relativity.* CUP, Cambridge, 2007.

[5] NewScientist.com news service, Mason Inman. Black holes may harbour their own universes. `http://space.newscientist.com/article/dn12853` 31 October 2007

[6] M. Bojowald, D. Cartin and G. Khanna. Lattice refining loop quantum cosmology, anisotropic models and stability. *Phys. Rev.*, D76:064018, 2007.

[7] A. Ashtekar and M. Bojowald. Quantum geometry and the Schwarzschild singularity. *Class. Quant. Grav.*, 23:391–411, 2006.

[8] L. Modesto. The Kantowski-Sachs space-time in loop quantum gravity. *Int. J. Theor. Phys*, 45:2235-2246, 2006.

[9] L. Modesto. Loop quantum black hole. *Class. Quant. Grav.*, 23:5587-5602, 2006.

[10] M. Campiglia, R. Gambini and J. Pullin. Loop quantization of spherically symmetric midi-superspaces: the interior problem. *AIP Conf. Proc.*, 977:52-63, 2008.

[11] R. Gambini and J. Pullin, Black holes in loop quantum gravity: the complete space-time. arXiv:0805.1187 [gr-qc].

[12] H. Nariai. On some static solutions of Einstein's gravitational field equations in a spherically symmetric case. *Gen. Rel. Grav.*, 31:951–961, 1999. Originally published in *The Science Reports of the Tohoku University* Series I, vol. **XXXIV**, No. 3 (1950), p. 160–167.

[13] H. Nariai. On a new cosmological solution of Einstein's field equations of gravitation. *Gen. Rel. Grav.*, 31:963–971, 1999. Originally published in *The Science Reports of the Tohoku University* Series I, vol. **XXXV**, No. 1 (1951), p. 46–57.

[14] P. H. Ginsparg and M. J. Perry. Semiclassical Perdurance Of De Sitter Space. *Nucl. Phys.*, B222: 245–268 1983.

[15] R. Bousso and S. W. Hawking. Pair creation of black holes during inflation. *Phys. Rev.*, D54: 6312–6322, 1996.

[16] R. Bousso. Charged Nariai black holes with a dilaton. *Phys. Rev.*, D55: 3614–3621, 1997.

[17] L. Modesto. Black hole interior from loop quantum gravity. arXiv:gr-qc/0611043.